

Cuntz-like Algebras

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Dedicated to Professor Marc Rieffel on his 60th birthday

Abstract

The usual crossed product construction which associates to the homeomorphism T of the locally compact space X the C^* -algebra $C^*(X, T)$ is extended to the case of a partial local homeomorphism T . For example, the Cuntz-Krieger algebras are the C^* -algebras of the one-sided Markov shifts. The generalizations of the Cuntz-Krieger algebras (graph algebras, algebras O_A where A is an infinite matrix) which have been introduced recently can also be described as C^* -algebras of Markov chains with countably many states. This is useful to obtain such properties of these algebras as nuclearity, simplicity or pure infiniteness. One also gives examples of strong Morita equivalences arising from dynamical systems equivalences.¹

1 Introduction.

Let us recall (with no respect for history) two striking results pertaining the rich interplay between ergodic theory and von Neumann algebras (we refer the reader to the survey [19] and the references thereof for details; at that time, the theory was essentially complete except the uniqueness of the hyperfinite III_1 factor). A generalization [11] of the Murray-von Neumann group measure construction associates to each discrete measured equivalence relation (with possibly a twist) a von Neumann algebra, which is injective (or amenable) if and only if the measured equivalence relation is amenable in the sense of Zimmer. This construction is particularly well-behaved for amenable von Neumann algebras. First, each amenable von Neumann algebra arises from a discrete measured equivalence relations. Second, this discrete measured equivalence relation is unique up to isomorphism. There is an important fact which underlies these developments and also makes the connection with classical ergodic theory, namely: amenable equivalence relations are singly generated [4].

Although the interplay between topological dynamics and C^* -algebras may be more elusive, the study of transformation group C^* -algebras has proved for

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1991 *Mathematics Subject Classification*. Primary: 46L55, 43A35 Secondary: 43A07, 43A15, 43A22.

Key words and phrases. Dynamical systems. C^* -algebras. Endomorphisms. Markov shifts. Morita equivalence.

over thirty years to be of great importance, both for the internal theory of C^* -algebras and for its applications.

In [3], we have proposed étale essentially principal groupoids as a topological analogue of discrete measured equivalence relations; just as discrete equivalence relations arise from group actions, these groupoids arise as groupoids of germs of pseudogroups. Via the groupoid construction of [21] (or the equivalent localization construction of [14]), one obtains a large class of C^* -algebras, which are nuclear if and only if the groupoid is amenable in the sense of [2]. We shall study here a subclass of these algebras, namely those arising from singly generated pseudogroups (a precise definition will be given below). We view the groupoids of germs of singly generated pseudogroups as a (sort of) topological analogue of singly generated equivalence relations; in particular, shifts and shift spaces will provide our basic examples.

In the first section, we shall introduce this class of “Cuntz-like” algebras and show that they share some of the features of the classical Cuntz algebras O_n . In the second section, we consider some groupoid equivalences, in particular those arising from shift equivalences. The results presented there are well known. In the third section, we shall describe as groupoid C^* -algebras the examples of the Cuntz-Krieger algebras for infinite matrices constructed by R. Exel and M. Lacas in [10]. These examples were a strong motivation for our definitions.

This work, which is an extension of a talk given at the 17th Conference on Operator Theory at Timisoara in June 98, illustrates the use of some groupoid techniques in the study of Cuntz-Krieger algebras. It only covers a limited part of the rich domain of the Cuntz-Krieger algebras and their generalizations.

I heartily thank Claire Anantharaman-Delaroche, Marcelo Lacas and Georges Skandalis for valuable comments.

2 Singly generated pseudogroups and their C^* -algebras.

We first recall some definitions from [3].

Definition 2.1. Let X be a topological space. A partial homeomorphism S of X is a homeomorphism from an open set $\text{dom}(S)$ of X onto an open set $\text{ran}(S)$ of X . A pseudogroup \mathcal{G} on X is a set of partial homeomorphisms of X such that

- (i) the identity map 1_X belongs to \mathcal{G} ,
- (ii) $S, T \in \mathcal{G} \Rightarrow ST = S \circ T \in \mathcal{G}$,
- (iii) $S \in \mathcal{G} \Rightarrow S^{-1} \in \mathcal{G}$.

The pseudogroup \mathcal{G} is said to be full if

- (iv) for every open set U of X , $S \in \mathcal{G} \Rightarrow S|_U = S 1_U \in \mathcal{G}$,

- (v) each partial homeomorphism S which belongs locally to \mathcal{G} , i.e. every $x \in \text{dom}(S)$ has an open neighborhood U_x such that the restriction of S to U_x belongs to \mathcal{G} , does belong to \mathcal{G} .

The conditions (iv) and (v) are usually included in the definition of a pseudogroup (cf. [12]). By analogy with the notion of full group in ergodic theory, we prefer to state them apart. We remark that every pseudogroup \mathcal{G} is contained in a smallest full pseudogroup $\overline{\mathcal{G}}$. We say that $\overline{\mathcal{G}}$ is the full pseudogroup of \mathcal{G} . There are several groupoids, related to the semidirect product of X by \mathcal{G} , which can be attached to the pseudogroup \mathcal{G} . We are chiefly interested in its groupoid of germs, which depend only on the full pseudogroup. Let us recall its definition.

Definition 2.2. We say that two partial homeomorphisms S, T have the same germ at $x \in \text{dom}(S) \cap \text{dom}(T)$ if S and T agree on an open neighborhood of x . The equivalence class is called the germ at x and denoted by $[Sx, S, x]$. The groupoid of germs $G = \text{Germ}(X, \mathcal{G})$ of the pseudogroup \mathcal{G} on the topological space X is the set of germs of \mathcal{G} :

$$G = \{[x, S, y], S \in \mathcal{G}, y \in \text{dom}(S), x = Sy\}$$

endowed with the groupoid structure defined by the maps $r, s : G \rightarrow X$ such that $r[x, S, y] = x, s[x, S, y] = y$, the product $[x, S, y][y, T, z] = [x, ST, z]$ and the inverse $[x, S, y]^{-1} = [y, S^{-1}, x]$ and the topology of germs with

$$\mathcal{U}(S; U) = \{[x, S, y] : y \in U\}$$

where $S \in \mathcal{G}$ and U is an open subset of $\text{dom}(S)$ as basic open sets.

If X is a Baire space, then the groupoid of germs $\text{Germ}(X, \mathcal{G})$ is essentially principal in the sense that the set of points with trivial isotropy is dense (cf. Proposition 2.1 of [3]).

Definition 2.3. We define a *singly generated dynamical system* (SGDS) as a pair (X, T) where X is a topological space and T is a local homeomorphism from an open subset $\text{dom}(T)$ of X onto an open subset $\text{ran}(T)$ of X . We denote by $\mathcal{G}(X, T)$ the full pseudogroup generated by the restrictions $T|_U$, where U is an open subset of X on which T is injective.

Lemma 2.1. Let (X, T) be a SGDS.

- (i) A partial homeomorphism S belongs to $\mathcal{G}(X, T)$ iff it is locally of the form $(T|_U^m)^{-1}T|_V^n$, where $m, n \in \mathbf{N}$, U is an open set on which T^m is injective and V is an open set on which T^n is injective.
- (ii) Let $x \in X$. Suppose that $(T|_U^m)^{-1}T|_V^n$ and $(T|_W^p)^{-1}T|_Y^q$ are two partial homeomorphisms as in (i) having x in their domains and sending it into the same element. If $m - n = p - q$, then $(T|_U^m)^{-1}T|_V^n$ and $(T|_W^p)^{-1}T|_Y^q$ have the same germ at x .

Proof. (i) It is clear that a partial homeomorphism of the above form $(T|_U^m)^{-1}T|_V^n$ is in $\mathcal{G}(X, T)$ and, since $\mathcal{G}(X, T)$ is full, this is still true for a partial homeomorphism locally of that form. On the other hand, the inverse of $(T|_U^m)^{-1}T|_V^n$, which is $(T|_V^n)^{-1}T|_U^m$, is of the same form. The product of $(T|_U^m)^{-1}T|_V^n$ and $(T|_W^p)^{-1}T|_Y^q$ is also of the same form: if $n \geq p$, it is of the form $(T|_U^m)^{-1}T|_Z^{n-p+q}$, where Z is an open set on which T^{n-p+q} is injective and if $n < p$, it is of the form $(T|_Z^{p-n+m})^{-1}T|_Y^q$, where Z is an open set on which T^{p-n+m} is injective. This implies that the set of partial homeomorphisms locally of the above form is a full pseudogroup, and therefore is $\mathcal{G}(X, T)$.

(ii) We may assume that $m - p = n - q = k \geq 0$, $W = U, Y = V$ and $T^p U = T^q V$. Then

$$(T|_U^m)^{-1}T|_V^n = (T|_U^p)^{-1}(T|_{T^p U}^k)^{-1}T|_{T^q V}^k T|_V^q = (T|_U^p)^{-1}T|_V^q.$$

□

We denote by $Germ(X, T)$ the groupoid of germs of $\mathcal{G}(X, T)$. We also consider another groupoid attached to (X, T) , the semidirect product groupoid:

Definition 2.4. Let (X, T) be a SGDS. Its semidirect product groupoid is

$$G(X, T) = \{(x, m - n, y) : m, n \in \mathbf{N}, x \in \text{dom}(T^m), y \in \text{dom}(T^n), T^m x = T^n y\}$$

with the groupoid structure induced by the product structure of the trivial groupoid $X \times X$ and of the group \mathbf{Z} and the topology defined by the basic open sets

$$\mathcal{U}(U; m, n; V) = \{(x, m - n, y) : (x, y) \in U \times V, T^m(x) = T^n(y)\}$$

where U [resp. V] is an open subset of the domain of T^m [resp. T^n] on which T^m [resp. T^n] is injective.

According to the above lemma, there is a map π from $G(X, T)$ onto $Germ(X, T)$ which sends $(x, m - n, y)$ into the germ $[x, (T|_U^m)^{-1}T|_V^n, y]$, where U is an open neighborhood of x on which T^m is injective and V is an open neighborhood of y on which T^n is injective. This map is continuous and is a groupoid homomorphism. Let us see when it is an isomorphism.

Definition 2.5. We shall say that a SGDS (X, T) is essentially free if for every pair of distinct integers (m, n) , there is no nonempty open set on which T^m and T^n agree.

Lemma 2.2. Let (X, T) be an essentially free SGDS. Then,

- (i) If $(T|_U^m)^{-1}T|_V^n$ and $(T|_W^p)^{-1}T|_Y^q$, where $m, n, p, q \in \mathbf{N}$ and U, V, W, Y are open sets such that $T|_U^m, T|_V^n, T|_W^p, T|_Y^q$ are injective, have the same germ at x , then $m - n = p - q$.

(ii) The map $c : \text{Germ}(X, T) \rightarrow \mathbf{Z}$ such that $c[(T|_U^m)^{-1}T|_V^n x, (T|_U^m)^{-1}T|_V^n x] = m - n$ is a continuous homomorphism.

Proof. (i) By assumption, the relation $T^{n+p}y = T^{m+q}y$ holds on a nonempty open set. The essential freeness implies that $n + p = m + q$.

(ii) We have seen in the previous proposition that the product of $(T|_U^m)^{-1}T|_V^n$ and $(T|_W^p)^{-1}T|_Y^q$ is of the form $(T|_U^m)^{-1}T|_Z^{n-p+q}$ or $(T|_Z^{p-n+m})^{-1}T|_Y^q$ and that the inverse of $(T|_U^m)^{-1}T|_V^n$ is $(T|_V^n)^{-1}T|_U^m$. This shows that c is a homomorphism. By construction, c is locally constant. \square

Proposition 2.3. *Let (X, T) be a SGDS. Then, (X, T) is essentially free if and only if the above surjection $\pi : G(X, T) \rightarrow \text{Germ}(X, T)$ is an isomorphism.*

Proof. If (X, T) is essentially free, we may define the map ρ from $\text{Germ}(X, T)$ to $G(X, T)$ by $\rho[y, S, x] = (y, c(y, S, x), x)$. This map is an inverse for π .

Conversely, suppose that π is an isomorphism. Let $m, n \in \mathbf{N}$ and U be a nonempty open set such that $T|_U^m = T|_U^n$. Then for $x \in U$, the germ $\pi(x, m - n, x) = [x, (T|_U^m)^{-1}T|_U^n, x]$ is a unit. This implies that $m = n$. \square

From now on, we shall assume that the space X of our SGDS (X, T) is Hausdorff, second countable and locally compact. Then $G(X, T)$ is a Hausdorff locally compact étale groupoid and we can construct its C^* -algebra $C^*(X, T) = C^*(G(X, T))$ (we shall soon see that we do not have to distinguish the full and the reduced C^* -algebras).

Here is an elementary example of a SGDS. We let X be the one-point compactification of $\mathbf{N} \cup \{\infty\}$ and $T : \mathbf{N} \rightarrow \mathbf{N}$ be the translation $i \mapsto i + 1$. Then (X, T) is essentially principal and $G(X, T) = \text{Germ}(X, T) = (\mathbf{N} \times \mathbf{N}) \cup \{(\infty, \infty)\}$ where the sets $\{(i, i) : i \geq i_0\}$ form a fundamental system of neighborhoods of $\{(\infty, \infty)\}$. Its C^* -algebra is the algebra obtained by adjoining a unit to the algebra of compact operators on $l^2(\mathbf{N})$.

An other example of SGDS is provided by the full one-sided shift, where $X = \{1, \dots, d\}^{\mathbf{N}}$, d is an integer, and $Tx_i = x_{i+1}$. Then $C^*(X, T)$ is the Cuntz algebra O_d (cf. Section III.2 of [21]). A generalization of this example will be discussed in the third section. Some of the features of the Cuntz algebras are kept in the general situation. The crossed product nature of $C^*(X, T)$ is revealed by the fundamental homomorphism $c : G(X, T) \rightarrow \mathbf{Z}$, which induces the dual action (cf. [21], Section II.5) α of $\mathbf{T} = \hat{\mathbf{Z}}$ on $C^*(X, T)$ according to $\alpha_t(f)(x, k, y) = e^{ikt}f(x, k, y)$ for $f \in C_c(G(X, T))$. The kernel of c will be denoted $R(X, T) = c^{-1}(0)$. It is an open and closed subgroupoid of $G(X, T)$ and it has no isotropy. It is reduced to the unit space X when T is a global homeomorphism of X onto X . Its C^* -algebra $C^*(R(X, T))$ is the fixed point algebra $C^*(X, T)^\alpha$. In the case of the full one-sided shift as above, $C^*(R(X, T))$ is the UHF algebra $UHF(d^\infty)$.

Proposition 2.4. *Let (X, T) be a SGDS. Then,*

(i) $G(X, T)$ is amenable;

(ii) the full and reduced C^* -algebras coincide;

(iii) the C^* -algebra $C^*(X, T)$ is nuclear.

Proof. (i) We will check measurewise amenability (according to [2], 3.3.7, it is equivalent to topological amenability for étale groupoids). We first show that $R(X, T) = c^{-1}(0)$ is amenable. It is the increasing union of $R_N = \{(x, y) \in X \times X : \exists n \leq N : x, y \in \text{dom}(T^n), T^n x = T^n y\}$. According to Section 5.f of [2], it suffices to show that R_N is amenable. This is a Borel equivalence relation with countable equivalence classes. There is a countable family of open sets U_i which covers X and such that, for each $n \in \{0, \dots, N\}$, the restrictions $T^n|_{U_i}$ are one-to-one. Therefore, the equivalence relation R_N is countably separated, its quotient space X/R_N is analytic . This implies that R_N is a proper Borel groupoid ([2], 2.1.2), hence it is amenable. We can now apply a result on the amenability of an extension ([2]; 5.2.13) to conclude that $G(X, T)$ is amenable. Indeed, we may write X as the disjoint union of the invariant Borel subsets Y and Z , where Z is the intersection of the domains of the T^n 's and Y is its complement. The homomorphism $c : G(X, T) \rightarrow \mathbf{Z}$ is strongly surjective in the sense given there on Z . On the other hand, the reduction of $G(X, T)$ to Y is a proper principal groupoid having as quotient space the complement U^c of the domain U .

(ii) This is a well-known property of the C^* -algebra of an amenable groupoid (see e.g. [2]; 6.1.5).

(iii) This is also a well-known property of the C^* -algebra of an amenable groupoid (see e.g. [2]; 6.2.14). \square

According to [22], since $G(X, T)$ is Hausdorff and amenable, the ideal structure of $C^*(X, T)$ is very much related to the structure of the open invariant subsets of X . Note that invariance with respect to a groupoid of germs G of a pseudogroup \mathcal{G} simply means invariance under \mathcal{G} . A sufficient condition ([22], 4.9) for an isomorphism between these two structures is that for every closed invariant subset of X , the reduced groupoid $G(X, T)|_F$ has a dense set of points without isotropy (a word of caution has to be given here: the reduced groupoid $G|_F$ is usually distinct from the groupoid of germs of the reduced pseudogroup $\mathcal{G}|_F$). In particular, we have the following criterion for the simplicity of $C^*(X, T)$:

Proposition 2.5. *Let (X, T) be an essentially free SGDS. Assume that for every nonempty open set $U \subset X$ and every $x \in X$, there exist $m, n \in \mathbf{N}$ such that $x \in \text{dom}(T^n)$ and $T^n x \in T^m U$. Then $C^*(X, T)$ is simple.*

We can also express in our particular case the locally contracting property of Section 2. 4 of [1] which ensures that the C^* -algebra is purely infinite.

Proposition 2.6. *Let (X, T) be an essentially free SGDS. Assume that for every nonempty open set $U \subset X$, there exist an open set $V \subset U$ and $m, n \in \mathbf{N}$ such that $T^n(V)$ is strictly contained in $T^m(V)$. Then $C^*(X, T)$ is purely infinite.*

3 Groupoid equivalence and shift equivalence.

3.1 Equivalence of pseudogroups

Definition 3.1. Let (X, \mathcal{G}) and (Y, \mathcal{H}) be two full pseudogroups. An isomorphism from \mathcal{G} to \mathcal{H} is a homeomorphism $\varphi : X \rightarrow Y$ such that $\varphi\mathcal{G}\varphi^{-1} = \mathcal{H}$.

It is useful to introduce a notion weaker than isomorphism of pseudogroups.

Definition 3.2. Let (X, \mathcal{G}) be a full pseudogroup and let A, B be open subsets of X . We denote by \mathcal{G}_A^B the set of elements $S \in \mathcal{G}$ which have their domains contained in B and their ranges contained in A . In particular, $\mathcal{G}_{|A} = \mathcal{G}_A^A$ is called the reduction of \mathcal{G} to A . We say that $A \subset X$ is full (with respect to \mathcal{G}) if it meets every orbit under \mathcal{G} .

Definition 3.3. (Cf. [12]) Let (X, \mathcal{G}) and (Y, \mathcal{H}) be two full pseudogroups. An equivalence from \mathcal{G} to \mathcal{H} is a maximal collection \mathcal{Z} of partial homeomorphisms from X to Y such that

- (i) The domains of the elements of \mathcal{Z} form an open cover of X and the ranges of the elements of \mathcal{Z} form an open cover of Y .
- (ii) $R \in \mathcal{G}, \varphi \in \mathcal{Z}, T \in \mathcal{H} \implies T\varphi R \in \mathcal{Z}$.
- (iii) $R \in \mathcal{G}, \varphi, \psi \in \mathcal{Z}, T \in \mathcal{H} \implies \psi R\varphi^{-1} \in \mathcal{G}, \psi^{-1}T\varphi \in \mathcal{H}$.

Let (X, \mathcal{G}) and (Y, \mathcal{H}) be two full pseudogroups and let \mathcal{Z} be an equivalence from \mathcal{G} to \mathcal{H} . Let \mathcal{I} be the full pseudogroup on the disjoint union $Z = X \sqcup Y$ generated by $\mathcal{G}, \mathcal{H}, \mathcal{Z}$. Then X and Y are full with respect to \mathcal{I} and $\mathcal{G} = \mathcal{I}_{|X}, \mathcal{H} = \mathcal{I}_{|Y}$. Conversely, let (Z, \mathcal{I}) be a full pseudogroup and X, Y two full open subsets of Z . Then \mathcal{I}_Y^X is an equivalence from $\mathcal{I}_{|X}$ onto $\mathcal{I}_{|Y}$.

Let (X, \mathcal{G}) and (Y, \mathcal{H}) be two pseudogroups. Every collection \mathcal{Z} of partial homeomorphisms from Y to X such that

- (i) The union of the domains of the elements of \mathcal{Z} meets every orbit under \mathcal{G} and the union of the ranges of the elements of \mathcal{Z} meets every orbit under \mathcal{H} .
- (ii) $S \in \mathcal{H}, \varphi, \psi \in \mathcal{Z}, T \in \mathcal{G} \implies \varphi T\psi^{-1} \in \overline{\mathcal{H}}, \varphi^{-1}S\psi \in \overline{\mathcal{G}}$

can be completed in a unique way into an equivalence $\overline{\mathcal{Z}}$ between $\overline{\mathcal{H}}$ and $\overline{\mathcal{G}}$. Explicitly, a partial homeomorphism φ from Y to X belongs to $\overline{\mathcal{Z}}$ iff for every point y of its domain, there exist an open neighborhood U of y , $S \in \mathcal{H}, \psi \in \mathcal{Z}, T \in \mathcal{G}$ such that $\varphi|_U = S\psi T|_U$. We then say that the collection \mathcal{Z} generates $\overline{\mathcal{Z}}$.

The proof of the following proposition is straightforward.

Proposition 3.1. Let (X, \mathcal{G}) and (Y, \mathcal{H}) be two full pseudogroups and let G and H be their groupoids of germs.

- (i) An isomorphism $\varphi : X \rightarrow Y$ from \mathcal{G} onto \mathcal{H} implements the groupoid isomorphism $\varphi : G \rightarrow H$ such that $\varphi[x, S, y] = [\varphi(x), \varphi S \varphi^{-1}, \varphi(y)]$.
- (ii) An equivalence \mathcal{Z} from \mathcal{G} onto \mathcal{H} implements the groupoid equivalence Z from G onto H , where Z is the space of germs of \mathcal{Z} .

Example 3.1. *The irrational rotations algebras.* (cf. [23] and [14]) Let a be an irrational number. Then the pseudogroup \mathcal{G} on $X = \mathbf{R}$ generated by the translations $x \mapsto x + 1$ and $x \mapsto x + a$ is equivalent to the pseudogroup \mathcal{H} on $Y = \mathbf{R}/\mathbf{Z}$ generated by the translation $y \mapsto y + a$. Indeed the restriction of the quotient map $X \rightarrow Y$ to a nonempty open interval of length strictly less than one generates an equivalence from \mathcal{G} onto \mathcal{H} . Therefore, if a and b are irrational numbers such that $\mathbf{Z} + a\mathbf{Z} = \mathbf{Z} + b\mathbf{Z}$, then the groupoids of the corresponding irrational rotations are equivalent and their C^* -algebras are strongly Morita equivalent.

3.2 Strong shift equivalence.

We shall limit ourselves in this section to SGDS (X, S) such that $\text{dom}(S) = \text{ran}(S) = X$. We will investigate further the structure of the semidirect product $G(X, S)$. We have introduced the fundamental homomorphism (or cocycle) $c : G(X, S) \rightarrow \mathbf{Z}$ such that $c(x, m - n, y) = m - n$ and its kernel $R(X, S)$. Another important piece of structure is the fundamental endomorphism of $G(X, S)$, which is still denoted by S and which is defined by $S(x, m - n, y) = (S(x), m - n, S(y))$. This is indeed a groupoid homomorphism of $G(X, S)$ into itself; it is surjective since $S : X \rightarrow X$ is assumed to be surjective; it preserves the fundamental cocycle in the sense that $c \circ S = c$ and induces an endomorphism of $R(X, S)$ onto itself. As an endomorphism of $G(X, S)$, it is equivalent to the identity since we have $S(\gamma) = (b \circ r(\gamma))^{-1} \gamma b \circ s(\gamma)$ where $b(x) = (x, 1, Sx)$. Its graph is a groupoid equivalence isomorphic to the identity equivalence. However, in general, it is no longer trivial as an endomorphism of $R(X, S)$.

Lemma 3.2. *The graph $Z(S)$ of the fundamental homomorphism is a groupoid equivalence from $R(X, S)$ onto itself.*

Proof. This graph is the space

$$Z(S) = \{(x, y) \in X \times X : \exists m : S^{m+1}(x) = S^m(y)\}$$

endowed with the natural left and right actions of $R(X, S)$. One can check directly the axioms of [20]. \square

Given the SGDS (X, S) , we form the SGDS (\tilde{X}, \tilde{S}) where \tilde{X} is the projective limit of the projective system

$$\dots X_2 \xrightarrow{\pi_{1,2}} X_1 \xrightarrow{\pi_{0,1}} X_0$$

where $X_i = X$ and $\pi_{i,i+1} = S$ for all $i = 0, 1, \dots$ and \tilde{S} is the homeomorphism induced by the projective system morphism $S_i = S : X_i \rightarrow X_i$. Then \tilde{S} is invertible and its inverse is defined by the projective system morphism $T_i = id : X_{i+1} \rightarrow X_i$. We denote by $\pi = \pi_0$ the projection $\tilde{X} \rightarrow X = X_0$ and call (\tilde{X}, \tilde{S}) the invertible extension of (X, S) .

Definition 3.4. Let (X, S) and (Y, T) be two SGDS. A (strong) shift equivalence of lag $k \in \mathbf{N}$, consists of a pair of continuous maps $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ such that

$$\begin{aligned} T\varphi &= \varphi S ; & \psi T &= S\psi \\ \psi\varphi &= S^k ; & \varphi\psi &= T^k. \end{aligned}$$

The definition implies that φ and ψ are surjective and local homeomorphisms. When S and T are invertible, φ and ψ are simply conjugacies. When S and T are not invertible, the strong shift equivalence (φ, ψ) from (X, S) to (Y, T) induces a pair of morphisms of the associated projective systems, hence a pair of continuous maps $(\tilde{\varphi}, \tilde{\psi})$ from (\tilde{X}, \tilde{S}) to (\tilde{Y}, \tilde{T}) which is also a strong shift equivalence with the same lag. Therefore, $\tilde{\varphi}$ and $\tilde{\psi}$ are conjugacies.

Let us look at the strong shift equivalence from the groupoid point of view. The condition $T\varphi = \varphi S$ implies that φ induces a groupoid homomorphism, still denoted by φ , from $G(X, S)$ onto $G(Y, T)$ such that $\varphi(x, m - n, y) = (\varphi(x), m - n, \varphi(y))$. Since it preserves the fundamental cocycles, it also defines a groupoid homomorphism from $R(X, S)$ onto $R(Y, T)$. Its graph is a groupoid equivalence which intertwines the fundamental equivalences $Z(S)$ and $Z(T)$.

At the level of the C^* -algebras, a strong shift equivalence implements a strong Morita equivalence between $C^*(X, S)$ and $C^*(Y, T)$ which is trivial in K-theory and a strong Morita equivalence between $C^*(R(X, S))$ and $C^*(R(Y, T))$ which induces an isomorphism of their K-groups which intertwines the automorphisms induced by S and T .

Example 3.2. Topological Markov Shifts. We first recall from [17] the definition of the edge shift associated with a graph. Let Γ be a directed graph with finite vertex set I and finite edge set E . The adjacency matrix of the graph is the nonnegative integer valued matrix A whose element $A(i, j)$ is the number of edges from vertex i to vertex j . Up to isomorphism Γ depends only on A and we write $\Gamma = \Gamma_A$. We denote by X_A the space of one-sided infinite paths $e_0 e_1 e_2 \dots$, where $e_n \in E$ and $r(e_n) = s(e_{n+1})$ endowed with the product topology and by T_A the one-sided shift $(T_A e)_n = e_{n+1}$ on X_A .

Recall that an elementary equivalence between two nonnegative integral square matrices A and B is a pair (R, S) of nonnegative integral matrices such that $A = RS$ and $B = SR$. A strong shift equivalence between A and B is a finite sequence of nonnegative integral square matrices $A_0 = A, A_1, \dots, A_k = B$ and elementary equivalences between A_{i-1} and A_i , for $i = 1, \dots, k$. A strong shift equivalence between the matrices A and B gives a strong shift equivalence between the SGDS (X_A, T_A) and (X_B, T_B) . It suffices to consider the case of an elementary equivalence (R, S) . One draws $R(i, j)$ edges from the vertex i

of Γ_A to the vertex j of Γ_B and call E_R this set of edges. One constructs similarly a set of edges E_S . The condition $A = RS$ gives the existence of a bijection α from E_A onto the set $E_R * E_S$ of paths rs with $r \in E_R, s \in E_S$ preserving the initial and terminal vertices. Similarly, one chooses a bijection β from E_B onto $E_S * E_R$. One defines $\varphi : X_A \rightarrow X_B$ by $\varphi(a_0 a_1 \dots) = b_0 b_1 \dots$ where $\alpha(a_0) = r_0 s_0, \alpha(a_1) = r_1 s_1, \dots$ and $\beta(b_0) = s_0 r_1, \beta(b_1) = s_1 r_2, \dots$. One defines similarly $\psi : X_B \rightarrow X_A$. One checks that φ and ψ are continuous and that $T_A = \psi\varphi$ and $T_B = \varphi\psi$. Therefore, according to the above, a strong shift equivalence between the matrices A and B gives a groupoid equivalence between the principal groupoids $R(X_A, T_A)$ and $R(X_B, T_B)$ which intertwines the fundamental equivalences $Z(T_A)$ and $Z(T_B)$. In this example, $C^*(R(X_A, T_A))$ is an AF-algebra having for dimension group the inductive limit

$$G(A) = \lim_{\rightarrow} (\mathbf{Z}^{n_A} \xrightarrow{A} \mathbf{Z}^{n_A} \xrightarrow{A} \dots)$$

and T_A induces the shift automorphism τ_A . Thus we have a realization at the groupoid level of the shift preserving isomorphisms of the dimension groups $G(A)$ and $G(B)$ induced by the strong shift equivalence.

On the other hand, recall that a shift equivalence of lag k between two nonnegative integral square matrices A and B is a pair (R, S) of nonnegative integral matrices such that

$$\begin{aligned} AR &= RB ; & SA &= BS \\ RS &= A^k ; & SR &= B^k. \end{aligned}$$

In particular (R, S) is an elementary equivalence between A^k and B^k . Note that (X_{A^k}, T_{A^k}) is conjugate to (X_A, T_A^k) and that $R(X_A, T_A^k) = R(X_A, T_A)$. Thus, if A and B are only shift equivalent, we can still construct a groupoid equivalence between $R(X_A, T_A)$ and $R(X_B, T_B)$. Because of the other two relations, it induces a shift preserving isomorphisms of the dimension groups $G(A)$ and $G(B)$ (in fact A and B are shift equivalent if and only if $(G(A), G(A)^+, \tau_A)$ and $(G(B), G(B)^+, \tau_B)$ are isomorphic [13]). This equivalence intertwines the equivalences $Z(T_A^k)$ and $Z(T_B^k)$ but not necessarily the equivalences $Z(T_A)$ and $Z(T_B)$.

4 Graphs and Cuntz-Krieger algebras.

We describe in this section a class of SGDS (X, T) which appears implicitly in the work [10] of R. Exel and M. Laca. The Cuntz-Krieger algebras for infinite matrices which they construct are the C^* -algebras of these SGDS.

Definition 4.1. Let X be a compact totally disconnected space, U, V two open subsets and let $T : U \rightarrow V$ be a local homeomorphism from U onto V . A Markov partition for (X, T) is a partition of U by a family $\{U_i, i \in I\}$ of nonempty pairwise disjoint compact open subsets such that

- (i) the restriction $T_i = T|_{U_i}$ is a homeomorphism from U_i onto a compact open subset $V_i = T(U_i)$;
- (ii) for all $(i, j) \in I \times I$, either $U_i \subset V_j$ or $U_i \cap V_j = \emptyset$;
- (iii) the Boolean algebra \mathcal{B}_0 generated by $\{X, U_i, V_i, i \in I\}$ is a generator in the sense that $\bigvee_{n=0}^{\infty} T^{-n}\mathcal{B}_0$ is the family of all compact open subsets of X .

Definition 4.2. A Markov shift is a SGDS (X, T) which admits a Markov partition and which has a dense domain.

The analysis of a SGDS (X, T) admitting a Markov partition $\{U_i, i \in I\}$ can be done entirely from the subset $A \subset I \times I$ defined by $(j, i) \in A \Leftrightarrow V_j \supset U_i$ and the subset $\mathcal{J} \subset \mathcal{P}(I)$ defined by

$$J \in \mathcal{J} \Leftrightarrow \bigcap_J V_j \cap_{I \setminus J} V_k^c \cap U^c \neq \emptyset.$$

Because of the condition (iii) of Definition 4.1, this intersection contains at most one point, which will be denoted by x_J for $J \in \mathcal{J}$.

Let us first determine the spectrum X_0 of the Boolean algebra \mathcal{B}_0 generated by $\{X, U_i, V_i, i \in I\}$. Given $x \in X$, either $x \in U$ and then there exists a unique i , which we denote by $i = i_0(x)$, such that $x \in U_i$ or $x \in U^c$ and then we set $I(x) = \{j \in I : x \in V_j\}$. Note that $I(x)$ belongs to \mathcal{J} and that $x \mapsto I(x)$ is a bijection from U^c onto \mathcal{J} , whose inverse map is $J \mapsto x_J$. Putting these two maps together, we get a bijection, denoted by σ_0 , from X_0 onto the disjoint union $I \sqcup \mathcal{J}$. It remains to describe its topology. On I , it is the discrete topology. On the other hand, \mathcal{J} is a closed subset of $\mathcal{P}(I) = 2^I$ endowed with the product topology and the map: σ_0 is a homeomorphism from U^c onto \mathcal{J} . We introduce the map $J : I \rightarrow \mathcal{P}(I)$ defined by $J(i) = \{j \in I : V_j \supset U_i\}$ and note that a sequence (x_λ) in U converges to $x \in U^c$ iff $(J(i_0(x_\lambda)))$ converges to $\sigma_0(x)$. This shows that \mathcal{J} contains the cluster points of the map J and that the topology of $I \sqcup \mathcal{J}$ turning the symbol map σ_0 into a homeomorphism is the topology induced by the compact space $I^+ \times \mathcal{P}(I)$, where I^+ is the one-point compactification of I and $I^+ \times \mathcal{P}(I)$ has the product topology and where $I \sqcup_J \mathcal{J}$ is identified with the closed subset $\text{Graph}(J) \cup \{\infty\} \times \mathcal{J}$. We denote by $\tilde{X}_0 = I \sqcup_J \mathcal{J}$ this topological space.

A similar analysis provides the spectrum X_n of the Boolean algebra $\mathcal{B}_n = \bigvee_{i=0}^n T^{-i}\mathcal{B}_0$: every $x \in X$ defines a character of \mathcal{B}_n : one considers its orbit (x, Tx, T^2x, \dots) . If $x \in T^{-n}(U)$, then x is coded by the sequence $\sigma_n(x) = (i_0(x), \dots, i_n(x))$ where $i_k(x) = i_0(T^k x)$. If not, there exists an exit time $\tau = \tau(x) \leq n$ such that $T^\tau x \notin U$; then x is coded by the sequence $\sigma_n(x) = (i_0(x), \dots, i_{\tau-1}(x); I(T^\tau x))$. Thus we obtain a bijection σ_n from X_n onto the disjoint union $A^{(n)} \sqcup Y_n \dots \sqcup Y_1 \sqcup \mathcal{J}$, where $A^{(m)} = \{(i_0, \dots, i_m) \in I^{m+1} : (i_k, i_{k+1}) \in A \text{ and } Y_r = \{(i_0, \dots, i_{r-1}; J) \in A^{(r-1)} \times \mathcal{J} : i_{r-1} \in J\}\}$. Let us describe the topology of this disjoint union. As before, $A^{(n)}$ has the discrete

topology and Y_r has the topology of $A^{(r-1)} \times \mathcal{J}$. We have a sequence of continuous maps

$$A^{(n)} \xrightarrow{J_n} A^{(n-1)} \times \mathcal{P}(I) \rightarrow \dots \rightarrow A^{(0)} \times \mathcal{P}(I) \xrightarrow{J_0} \mathcal{P}(I)$$

where $J_m(i_0, \dots, i_m; J) = (i_0, \dots, i_{m-1}; J(i_m))$. We view this disjoint union as a closed subset of $A^{(n)+} \times (A^{(n-1)} \times \mathcal{P}(I))^+ \dots \times (A^{(0)} \times \mathcal{P}(I))^+ \times \mathcal{P}(I)$, where $x \in A^n$ is sent onto $(x, J_n(x), \dots, J_0 \circ J_1 \circ \dots \circ J_n(x))$ and $x \in Y^r$ is sent onto $(\infty, \dots, \infty, x, J_{r-1}(x), \dots, J_0 \circ \dots \circ J_{r-1}(x))$. This topological space will be denoted by $\tilde{X}_n = A^{(n)} \sqcup_{J_n} Y_n \dots \sqcup_{J_1} Y_1 \sqcup_{J_0} \mathcal{J}$.

One deduces the following description of the spectrum X of the Boolean algebra $\mathcal{B} = \cup_{n=0}^{\infty} \mathcal{B}_n$. We define $X_{A,\mathcal{J}}$ as the set of terminal paths, where a terminal path is either an infinite path (i_0, i_1, \dots) or a pair $(i_0 \dots i_n, J)$ consisting of a finite path $i_0 \dots i_n$ and a set $J \in \mathcal{J}$ containing i_n . The empty path \emptyset is considered as finite path of length 0. The corresponding terminal paths are (\emptyset, J) with $J \in \mathcal{J}$. We note that $X_{A,\mathcal{J}}$ is the projective limit of the \tilde{X}_n 's, where the projection $\pi_{n,n+1} : \tilde{X}_{n+1} \rightarrow \tilde{X}_n$ sends $(i_0, \dots, i_n, i_{n+1}) \in A^{(n+1)}$ and $(i_0, \dots, i_n; J) \in Y_{n+1}$ onto $(i_0, \dots, i_n) \in A^n$ and is the identity map elsewhere and we endow it with the projective limit topology. We define the symbol map $\sigma : X \rightarrow X_{A,\mathcal{J}}$ by looking at the orbit (x, Tx, T^2x, \dots) of $x \in X$. We call exit time $\tau = \tau(x)$ the smallest integer, if it exists, such that $T^\tau x \notin U$; if it does not exist, we set $\tau(x) = \infty$. For $n < \tau(x)$, we define $i_n(x) = i_0(T^n x)$. If x has infinite exit time, we define $\sigma(x)$ as the infinite path $(i_0(x), i_1(x), \dots)$ and if x has a finite exit time τ , we define $\sigma(x)$ as the terminal path $(i_0(x), \dots, i_{\tau-1}(x); I(T^\tau x))$. We note that the symbol maps σ_n define an isomorphism of the projective systems (X_n) and (\tilde{X}_n) and that σ is their limit. Therefore, it is a homeomorphism. The symbol map σ conjugates T and the one-sided shift $T_{A,\mathcal{J}}$ on $X_{A,\mathcal{J}}$, defined on the set $\tilde{U} = \sigma(U)$ of terminal paths of length ≥ 1 by $T_{A,\mathcal{J}}(i_0, \alpha) = \alpha$.

We summarize the above discussion.

Proposition 4.1. *Let I be a countable set, A a subset of $I \times I$ and \mathcal{J} a subset of $\mathcal{P}(I)$ containing the set \mathcal{J}_A of cluster points of the net $(A_i = \{j : (j, i) \in A\}), i \in I$. Assume that for each $i \in I$, either there exists $j \in I$ such that $(i, j) \in A$ or there exists $J \in \mathcal{P}(I)$ such that $i \in J$. Then the SGDS $(X_{A,\mathcal{J}}, T_{A,\mathcal{J}})$ admits the Markov partition $\{\tilde{U}_i\}, i \in I$, where \tilde{U}_i is the set of terminal paths starting with i .*

Moreover, any SGDS (X, T) which admits a Markov partition $\{U_i\}, i \in I$ is conjugate to the above model, where $A = \{(j, i) \in I \times I : V_j \supset U_i\}$ and $\mathcal{J} = \{J \in \mathcal{P}(I) : \cap_J V_j \cap_{I \setminus J} V_k^c \cap U^c \neq \emptyset\}$.

We are chiefly interested in Markov shifts, i.e. SGDS (X, T) admitting a Markov partition and having a dense domain.

Proposition 4.2. *With the notation of the above proposition, the domain $U = \cup_i U_i$ of the SGDS $(X_{A,\mathcal{J}}, T_{A,\mathcal{J}})$ is dense iff $\mathcal{J} = \mathcal{J}_A$.*

Proof. This is clear since, keeping the notation of the above discussion, $x \in U^c$ is in the closure of U iff $\sigma_0(x) \in \mathcal{J}_A$. \square

In this case, we shall simply write $X_A = X_{A,\mathcal{J}}$ and $T_A = T_{A,\mathcal{J}}$. Note that in this case the assumption of the part (i) of the above proposition is simply that for each $i \in I$, $A^i = \{j \in I : (i, j) \in A\}$ is nonempty. When we view I as the set of vertices and A as the set of edges of an oriented graph, this condition means that every vertex has at least one outgoing edge. When we view A as a matrix, this means that it has no zero rows.

The above analysis is also valid when I is finite. In this case \mathcal{J}_A is empty. If one chooses $\mathcal{J} = \emptyset$, one gets the usual Markov shift (X_A, T_A) . If one chooses $\mathcal{J} = \{I\}$, one gets the shift on the space of all sequences (finite or infinite) which gives the Toeplitz extension.

Exel and Laca give a necessary and sufficient condition on the graph $A \subset I \times I$ ensuring that the Markov shift (X_A, T_A) is essentially principal ([10], Proposition 12.2). This condition also appears in [16] in the case of a row-finite directed graph, where it is called exit condition (L). A loop of length $n \geq 1$ is a finite path $\alpha = (i_0, \dots, i_n = i_0)$. An edge (i_k, j) , where $j \neq i_{k+1}$ (where $i_{k+n} = i_k$) is called an outgoing edge of the loop α . A point $x \in X$ is called periodic with respect to the SGDS (X, T) if there exist $m < n$ such that x belongs to the domain of T^n and $T^m x = T^n x$. Then, we say that $n - m$ is a period of x . The periodic points of the Markov shift (X_A, T_A) are the infinite paths which, after some time, repeat indefinitely the same loop $\alpha = (i_0, \dots, i_p = i_0)$ and such an infinite path is isolated if and only if the loop α has no outgoing edge. Thus,

Proposition 4.3. *Let (X_A, T_A) be the Markov shift constructed from the graph $A \subset I \times I$ such that every vertex has at least one outgoing edge. The following conditions are equivalent:*

- (i) (X_A, T_A) is essentially principal;
- (ii) condition (L): every loop has at least one outgoing edge;
- (iii) (X_A, T_A) has no isolated periodic point.

Thus, when the graph $A \subset I \times I$ satisfies these conditions, the groupoid of germs $\text{Germ}(X_A, T_A)$ coincide with the semidirect product $G(X_A, T_A)$.

Let us study next the relation between a set $\{S_i, i \in I\}$ of partial isometries on a Hilbert space \mathcal{H} satisfying the Cuntz-Krieger relations associated to $A \subset I \times I$ and the C^* -algebra $C^*(X_A, T_A)$. In order to avoid infinite sums, Exel and Laca have introduced the following version of the Cuntz-Krieger relations, where $P_i = S_i S_i^*$, $Q_i = S_i^* S_i$:

- (CK 1) the Q_i 's commute;
- (CK 2) the P_i 's are pairwise orthogonal;
- (CK 3) $P_j Q_i = A(i, j) P_j$ for all $(i, j) \in I \times I$, where A is identified with its characteristic function;

- (CK 4) $\prod_{j \in E} Q_j \prod_{k \in F} (1 - Q_k) = \sum_{i \in I} A(E, F, i) P_i$ for all finite subsets $E, F \subset I$ such that $A(E, F, i) = \prod_{j \in E} A(j, i) \prod_{k \in F} (1 - A(k, i))$ is nonzero except for a finite number of i 's.

Let (X, T) be a SGDS admitting the Markov partition $\{U_i, i \in I\}$. We define $A \subset I \times I$ by $(i, j) \in A$ iff $U_j \subset V_i = T(U_i)$. For each $i \in I$, we have the bisection of $G(X, T)$:

$$S_i = \{(x, 1, Tx), x \in U_i\}.$$

By definition, the ranges $U_i = r(S_i)$ and the domains $V_i = s(S_i)$ of the S_i 's satisfy:

- (CK' 2) the U_i are pairwise disjoint;
- (CK' 3) for all $(i, j) \in I \times I$, $U_j \subset V_i$ or $U_j \subset V_i^c$ according to $A(i, j) = 1$ or $A(i, j) = 0$.

The following lemma elucidates the meaning of the condition (CK4).

Lemma 4.4. *Let (X, T) be a SGDS admitting a Markov partition $\{U_i, i \in I\}$. Then, its domain $U = \cup_I U_i$ is dense iff*

- (CK' 4) *whenever the intersection $\cap_F V_j \cap_G V_k^c$, where F, G are finite subsets of I , contains finitely many U_i 's, it is the union of these U_i 's.*

Proof. Suppose that U is dense. If the intersection $\cap_F V_j \cap_G V_k^c$, where F, G are finite subsets of I , contains only a finite union $\cap_E U_i$, then the complement of $\cap_E U_i$ in $\cap_F V_j \cap_G V_k^c$, which is open, must be empty. On the other hand, suppose that (iv') holds. We shall show that $\mathcal{J} = \mathcal{J}_A$, which by Proposition 4.2 is equivalent to the density of U . If $J \in \mathcal{P}(I)$ is not a cluster point of the map $i \mapsto J(i)$ introduced in the discussion following Definition 4.2, there exist finite subsets F, G of I such that $F \subset J, G \subset J^c$ and the conditions $F \subset J(i), G \subset J(i)^c$ are satisfied for only a finite number of i 's. This means that $\cap_F V_j \cap_G V_k^c$ contains only finitely many U_i 's; by (iv'), it is contained in U . By definition, J does not belong to \mathcal{J} . \square

Proposition 4.5. *Let I be a countable set and let A be a subset of $I \times I$.*

- (i) *Let \mathcal{J} be a subset of $\mathcal{P}(I)$ containing the set \mathcal{J}_A . Then the family of bisections $(S_i), i \in I$ of $G(X_{A, \mathcal{J}}, T_{A, \mathcal{J}})$, viewed as elements of $C^*(X_{A, \mathcal{J}}, T_{A, \mathcal{J}})$ is a family of partial isometries which satisfies the Cuntz-Krieger relations (CK1 – 3) relative to the matrix A . In particular, every representation of $C^*(X_{A, \mathcal{J}}, T_{A, \mathcal{J}})$ on a Hilbert space \mathcal{H} provides a representation of these relations.*
- (ii) *Conversely, let I be a countable set, let A be a $(0, 1)$ -matrix with no zero rows and let $(\underline{S}_i), i \in I$ be a family of nonzero partial isometries on a Hilbert space \mathcal{H} satisfying the Cuntz-Krieger relations (CK1 – 3)*

relative to the matrix A . Then, there exists a unique representation of $C^*(X_{A,\mathcal{J}}, T_{A,\mathcal{J}})$, where

$$\mathcal{J} = \{J \in \mathcal{P}(I) : E, F \text{ finite } E \subset J, F \subset J^c \Rightarrow \prod_E Q_j \prod_F (1 - Q_j) \neq 0\}.$$

on \mathcal{H} sending the bisection S_i into \underline{S}_i for each i .

Proof. (Cf. Proposition III.2.7 of [21] and Theorem 4.2 of [15]) For (i), the condition (CK1) is satisfied by construction. The conditions (CK2 – 3) are a restatement of above properties (CK'2 – 3).

For (ii), we give the sketch of the proof and refer to [21], [10] and [15] for details. Let $(\underline{S}_i), i \in I$ be such a family of partial isometries. A direct computation (see Proposition 3.2 of [10]) shows that the \underline{S}_i 's (together with 1) generate an inverse semigroup $\underline{\mathcal{S}}$ of partial isometries. Let X be the spectrum of the (commutative) C^* -algebra \mathcal{B} generated by the idempotents of \mathcal{S} and let $M : C(X) \rightarrow \mathcal{B}$ be the Gelfand isomorphism. For each i , let $U_i \subset X$ [resp. $V_i \subset X$] be the support of \underline{P}_i [resp. \underline{Q}_i]. The isomorphism $B \mapsto \underline{S}_i B \underline{S}_i^*$ from $\underline{Q}_i \mathcal{B}$ onto $\underline{P}_i \mathcal{B}$ induces a homeomorphism $T_i : U_i \rightarrow V_i$ and we define T on the (disjoint) union $U = \cup_i U_i$ by $Tx = T_i x$ if $x \in U_i$. Then $\{U_i, i \in I\}$ is a Markov partition for (X, T) with matrix A . According to Proposition 4.1, (X, T) is isomorphic to $(X_{A,\mathcal{J}}, T_{A,\mathcal{J}})$. Let $\{S_i, i \in I\}$ be the associated family of bisections of $G(X, T)$. It satisfies the same relations (CK1 – 3) as the family $\{\underline{S}_i, i \in I\}$. Let \mathcal{S} be the inverse semigroup of bisections of $G(X, T)$ generated by the S_i 's (and X). Given two finite paths $\alpha = (i_1, \dots, i_m)$ and $\beta = (j_1, \dots, j_n)$ and an idempotent h of \mathcal{S} , we define

$$S(\alpha, h, \beta) = S_{i_m}^{-1} \dots S_{i_1}^{-1} h S_{j_1} \dots S_{j_n} \quad \text{and}$$

$$\underline{S}(\alpha, h, \beta) = \underline{S}_{i_m}^* \dots \underline{S}_{i_1}^* M(h) \underline{S}_{j_1} \dots \underline{S}_{j_n}.$$

One checks that every element of \mathcal{S} can be written under the form $S(\alpha, h, \beta)$ and that this form is unique when we require h and the lengths of α, β to be minimal. The same is true for $\underline{\mathcal{S}}$. Therefore the map $S(\alpha, h, \beta) \mapsto \underline{S}(\alpha, h, \beta)$ is a representation L of the inverse semigroup \mathcal{S} onto $\underline{\mathcal{S}}$. This representation extends to a representation of $C^*(G(T))$. Indeed, every $f \in C_c(G(T))$ can be written as a finite sum $f = \sum (h_\alpha \circ r) S_\alpha$, where $h_\alpha \in C(X)$ and S_α is a compact open bisection. One can check that $L(f) = \sum M(h_\alpha) L(S_\alpha)$ is well defined and that L is a representation of the $*$ -algebra $C_c(G(T))$ continuous for the inductive limit topology and therefore extends to $C^*(G(T))$. \square

Corollary 4.6. *Let I be a countable set and let A be a subset of $I \times I$.*

- (i) *The family of bisections $(S_i), i \in I$ of $G(X_A, T_A)$, viewed as elements of $C^*(X_A, T_A)$ is a family of partial isometries which satisfies the Cuntz-Krieger relations (CK1 – 4) relative to the matrix A . In particular, every representation of $C^*(X_A, T_A)$ on a Hilbert space \mathcal{H} provides a representation of these relations.*

- (ii) Conversely, let I be a countable set, let A be a $(0,1)$ -matrix with no zero rows and let $(\underline{S}_i), i \in I$ be a family of nonzero partial isometries on a Hilbert space \mathcal{H} satisfying the Cuntz-Krieger relations (CK1 – 4) relative to the matrix A . Then, there exists a unique representation of $C^*(X_A, T_A)$, on \mathcal{H} sending the bisection S_i into \underline{S}_i for each i .

Proof. We have seen indeed the equivalence of the conditions (CK4) and $\mathcal{J} = \mathcal{J}_A$. \square

Because of this corollary, the C^* -algebra $C^*(X_A, T_A)$, where $A \subset I \times I$ has no zero rows, can be called the Cuntz-Krieger algebra of the matrix A . If moreover A the equivalent conditions of Proposition 4.3, we have the following uniqueness result.

Corollary 4.7. (Cf. [10], 13.2) Let $A \subset I \times I$ without zero rows and such that each loop has an outgoing edge. Let $\{S_i), i \in I\}$ and $\{T_i), i \in I\}$ be two families of nonzero partial isometries on Hilbert spaces \mathcal{H} and \mathcal{K} satisfying the Cuntz-Krieger relations (CK1 – 4) relative to the matrix A . Then there is an isomorphism from the C^* -algebra generated by $\{S_i), i \in I\}$ onto the C^* -algebra generated by $\{T_i), i \in I\}$ carrying S_i into T_i for all $i \in I$.

Proof. It suffices to show that the representation L of $C^*(X_A, T_A)$ defined by $\{S_i), i \in I\}$ is faithful. Since its restriction M to $C(X_A)$ is faithful and the groupoid $G(X_A, T_A)$ is nuclear and essentially principal, we may apply Theorem 4.3 of [22] to conclude. \square

By construction, $C^*(X_A, T_A)$ is unital and generated by the partial isometries S_i 's and 1. Exel and Laca distinguish this unital C^* -algebra, which they call \tilde{O}_A and the sub- C^* -algebra O_A generated by the S_i 's alone. There are two cases: either $\tilde{O}_A = O_A$ or \tilde{O}_A is the one-point compactification of O_A . Recall that in the description of the spectrum $X = X_A$, we have introduced the map $\sigma_0 : X \rightarrow X_0 = I \sqcup \mathcal{J}$, which is a homeomorphism from U^c onto \mathcal{J} . Its inverse is the map $J \mapsto (\emptyset; J)$ where the first component means the empty word. The complement of $U \cup V$ contains at most one point, namely the point $(\emptyset; \emptyset)$. The spectrum of the C^* -algebra generated by the P_i 's and the Q_i 's alone is either X_A or $X_A \setminus \{(\emptyset; \emptyset)\}$ according to $\emptyset \notin \mathcal{J}_A$ or $\emptyset \in \mathcal{J}_A$. Thus we have:

Proposition 4.8. (Cf. [10], 8.5) Let A be a $(0,1)$ -matrix with no zero rows.

- (i) if $\emptyset \notin \mathcal{J}_A$, then $O_A = C^*(X_A, T_A)$;
- (ii) if $\emptyset \in \mathcal{J}_A$, then $O_A = C^*(X_A \setminus \{(\emptyset; \emptyset)\}, T_A)$.

We may apply to the SGDS (X_A, T_A) the general criteria for simplicity and pure infiniteness given in the first section.

Proposition 4.9. (Cf. [10], 14.1) Assume that the matrix $A \subset I \times I$ satisfies (L) and is irreducible, in the sense that for each pair $(i, j) \in I \times I$, there is a finite path starting at i and ending at j . Then O_A is simple.

Proof. The SGDS (X_A, T_A) and $(X_A \setminus \{(\emptyset, \emptyset)\}, T_A)$ are essentially free and we may apply Proposition 2.5. Every nonempty open set W contains a cylinder set, i.e. a set $Z(\alpha)$ of all terminal paths starting with the finite path $\alpha = (j_0, \dots, j_n)$. Let $x \in X_A$ be an infinite path $x = (i_0, i_1, \dots)$. There exist a finite path $\beta = (j_n, \dots, j_{n+k} = i_0)$. Then $y = (j_0, \dots, j_n, \dots, j_{n+k} = i_0, i_1, \dots)$ belongs to $Z(\alpha)$, hence to W , and $T^{n+k}y = x$. Let x be a finite terminal path distinct from $(\emptyset; \emptyset)$. There exists an integer m such that $T^m x = (\emptyset; J)$ where J is nonempty. We choose $i \in J$ and a finite path $\beta = (j_n, \dots, j_{n+k} = i)$. Then $y = (j_0, \dots, j_n, \dots, j_{n+k} = i; J)$ belongs to W and $T^{n+k}y = T^m x$. \square

Proposition 4.10. (*Cf. [16], 3.9 and [10], 16.2*) Assume that the matrix $A \subset I \times I$ satisfies (L) and for every vertex $i \in I$, there exists a loop $(j_0, \dots, j_n = j_0)$ and a finite path starting at i and ending at some j_k . Then $C^*(X_A, T_A)$ and O_A are purely infinite.

Proof. We apply Proposition 2.6 to the SGDS (X_A, T_A) . First observe that if the loop $(j_0, \dots, j_n = j_0)$ has an outgoing edge, then $Z(j_0, \dots, j_n)$ is a proper subset of $T^n Z(j_0, \dots, j_n)$. We have seen that every nonempty open set W contains a cylinder set $Z(i_0, \dots, i_m)$. By assumption, there exists a loop $(j_0, \dots, j_n = j_0)$ and a finite path $i_m, \dots, i_{m+k} = j_0$. Let $V = Z(i_0, \dots, i_{m+k} = j_0, \dots, j_n)$. Then $V \subset W$ and $T^{m+k}V$ is a proper subset of $T^{m+k+n}V$. \square

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